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On States on the Product of Logics

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We take up the question of when a state ($=\sigma$-additive measure) on the product of logics ($=\sigma$-orthomodular posets) depends on at most countably many coordinates. We show that it is always so provided there are no real-measurable cardinals. The manner of dependence is a kind of convex combination. We derive some consequences of the latter statement.

It is commonly accepted that the set of all statements on a quantum mechanical system can be represented by a σ-orthomodular partially ordered set that is called a logic of the system (cf. Varadarajan, 1968). The states of the system then correspond to the σ-additive measures on the logic. It is assumed in some quantum system theories that the logic of a collection of quantum mechanical systems can be identified with the product of the logics of the individual systems (cf. Beltrametti and Cassinelli, 1976; von Neumann, 1955). The following question then appears naturally: Are the states on the product determined by the "coordinate states"? We show in this note that it is indeed so provided the cardinality of that collection of logics is not "very big." It should be noted that the particular case concerning two logics has been examined in the paper by Gudder (1966).

Let us first recall a few basic definitions (cf. Varadarajan, 1968; Gudder, 1966; etc.).

Definition 1. A logic ($=\sigma$-orthomodular poset in algebraic language) is a set L endowed with a partial ordering $\leqslant$ and a unitary operation $'$ such that

(i) $0, 1 \in L$

(ii) $a \leqslant b \Rightarrow b' \leqslant a'$ for any $a, b \in L$

(iii) $(a')' = a$ for any $a \in L$

(iv) $a \vee a' = 1$ for any $a \in L$

(v) $\bigvee_{n \in N} a_n$ exists in L whenever $a_n \in L$, $a_n \leqslant a_k'$ for $n \neq k$

(vi) $b = a \vee (b \wedge a')$ whenever $a, b \in L$, $a \leqslant b$

A subset K of a logic L is called orthogonal if $a \leqslant b'$ for any pair a, $b \in K$.

Definition 2. Let $\{L_\alpha | \alpha \in I\}$ be a collection of logics. Denote by $\Pi_{\alpha \in I} L_\alpha$ the ordinary Cartesian product of the sets L_α and endow $\Pi_{\alpha \in I} L_\alpha$ canonically with the relation $\prec$ and the unitary operation $'$. That is, if $k = \{k_1, k_2, \ldots\} \in \Pi_{\alpha \in I} L_\alpha$ and $h = \{h_1, h_2, \ldots\} \in \Pi_{\alpha \in I} L_\alpha$ then $k \prec h$ (respectively, $k' = h$) iff $k_\alpha \leqslant h_\alpha$ (respectively, $k'_\alpha = h_\alpha$) for any $\alpha \in I$. The set $\Pi_{\alpha \in I} L_\alpha$ (with $\prec$ and $'$) is called the product of the collection $\{L_\alpha | \alpha \in I\}$.

If $k \in \Pi_{\alpha \in I} L_\alpha$ then $k_\alpha \in L_\alpha$ denotes the αth coordinate of k.

Proposition 1. Let $\{L_\alpha | \alpha \in I\}$ be a collection of logics. Then $\Pi_{\alpha \in I} L_\alpha$ is a logic.

The proof is evident.

Definition 3. A state m on a logic L is a mapping $m\colon L \to \langle 0, 1 \rangle$ satisfying the following properties:

(i) $m(1) = 1$

(ii) if $\{a_n | n \in N\}$ is an orthogonal sequence of elements of L then

$$m\left(\bigvee_{n \in N} a_n \right) = \sum_{n=1}^{\infty} m(a_n)$$

We denote by $\mathbb{S}(L)$ the set of all states on L. We are going to discuss how (and when) the states on the product of logics are induced by "coordinate states." The point of departure is the following observation.

> *Theorem 1.* Let $\{L_\alpha | \alpha \in I\}$ be a collection of logics. Let $\{\alpha_n | n \in N\}$ be a countable subset of I and $\{p_{\alpha_n} | n \in N\}$ be a partition of unity (i.e., $p_{\alpha_n} \geqslant 0$ for any $n \in N$ and $\sum_{n=1}^{\infty} p_{\alpha_n} = 1$). If $\{m_{\alpha_n} | n \in N\}$ is a collection of states, $m_{\alpha_n} \in \mathbb{S}(L_{\alpha_n})$, then the mapping defined by the formula
>
> $$m(e_1, e_2, \ldots) = \sum_{n=1}^{\infty} p_{\alpha_n} \cdot m_{\alpha_n}(e_{\alpha_n})$$
>
> is a state on $\Pi_{\alpha \in I} L_\alpha$.

Proof. Evidently, $m(1) = m(1, 1, \ldots) = 1$. Suppose that $\{a_n | n \in N\}$ is an orthogonal sequence of elements of $\Pi_{\alpha \in I} L_\alpha$. We are to prove that

$$m\left(\bigvee_{n=1}^{\infty} a_n \right) = \sum_{n=1}^{\infty} m(a_n)$$

We may write $m=m_1\circ\varphi$, where φ is the projection of $\Pi_{\alpha\in I}L_\alpha$ onto $\Pi_{n\in N}L_{\alpha_n}$ and m_1 is the restriction of m to $\Pi_{n\in N}L_{\alpha_n}$. Since φ only forgets coordinates we obtain that $\{\varphi(a_n)|n\in N\}$ remains an orthogonal family and moreover, $\varphi(\vee_{n=1}^{\infty}a_n)=\vee_{n=1}^{\infty}\varphi(a_n)$. Put $b_n=\varphi(a_n)$, $b=\vee_{n=1}^{\infty}b_n$ and consider the value of $m(\vee_{n=1}^{\infty}a_n)$. We have

$$\begin{aligned}
m\left(\bigvee_{n=1}^{\infty}a_n\right)&=m_1\circ\varphi\left(\bigvee_{n=1}^{\infty}a_n\right)=m_1\left[\bigvee_{n=1}^{\infty}\varphi(a_n)\right]=m_1(b)\\
&=\sum_{n=1}^{\infty}p_{\alpha_n}\cdot m_{\alpha_n}(b_{\alpha_n})=\sum_{n=1}^{\infty}p_{\alpha_n}\cdot m_{\alpha_n}\left[\bigvee_{k=1}^{\infty}(b_k)_{\alpha_n}\right]\\
&=\sum_{n=1}^{\infty}p_{\alpha_n}\cdot\left\{\sum_{k=1}^{\infty}m_{\alpha_n}[(b_k)_{\alpha_n}]\right\}\\
&=\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty}p_{\alpha_n}\cdot m_{\alpha_n}[(b_k)_{\alpha_n}]\right\}\\
&=\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}p_{\alpha_n}\cdot m_{\alpha_n}[(b_k)_{\alpha_n}]\right\}\\
&=\sum_{k=1}^{\infty}m_1(b_k)=\sum_{k=1}^{\infty}m(a_k)=\sum_{n=1}^{\infty}m(a_n)
\end{aligned}$$

and the proof is finished. ■

Now, a question naturally arises if any state on the product can be expressed in the manner indicated above. The question turns out to be related to a set-theoretic assumption of the existence of real-measurable sets.

Definition 4. Let I be a set, let the symbol $\exp I$ denote the collection of all subsets of I. We say that I is non-real-measurable if there is no σ-additive measure μ: $\exp I\to\langle 0,1\rangle$ such that

(i) $\mu(I)=1$

(ii) $\mu\{\alpha\}=0$ for any $\alpha\in I$

Remark 1. The definition is a classical one and it is due to Ulam (1930). One can see easily that the set N of natural numbers is non-real-measurable. It is known (cf. Ulam, 1930) that if I is non-real-measurable then 2^I is also non-real-measurable. Further, if J_α is a non-real-measurable set for any $\alpha\in I$, I non-real-measurable, then $\cup_{\alpha\in I}J_\alpha$ is non-real-measurable as well. The latter propositions show that if real-measurable sets exist, which is not known within the ordinary set theories, they must be extremely big. It seems therefore harmless to assume that all sets are non-real-measurable as soon as we are in the realm of quantum system theories.

Theorem 2. Let $\{L_\alpha|\alpha\in I\}$ be a collection of logics, I non-real-measurable. Suppose m is a state on $\Pi_{\alpha\in I}L_\alpha$. Then we can find a sequence $\{\alpha_n|n\in N\}\subset I$, states $m_{\alpha_n}\in L_{\alpha_n}$ and a partition of unity $\{p_{\alpha_n}|n\in N\}$ such that

$$m(e)=m(e_1,e_2,\ldots)=\sum_{n=1}^{\infty} p_{\alpha_n}\cdot m_{\alpha_n}(e_{\alpha_n})$$

for any $e\in\Pi_{\alpha\in I}L_\alpha$.

Proof. Denote by e^α the element of $\Pi_{\alpha\in I}L_\alpha$ whose all but αth coordinates are zero and whose αth coordinate is e. Consider the set $M=\{\alpha\in I|m(1^\alpha)>0\}$. We see firstly that the set of M is at most countable. Indeed, if it were not the case we could find a countable subset M' of M such that $m(1^\alpha)\geqslant\varepsilon$ for an $\varepsilon>0$ and each $\alpha\in M'$, which is clearly absurd. We claim further that $\Sigma_{\alpha\in M}m(1^\alpha)=1$. Suppose the contrary. Then $m(1\wedge(\vee_{\alpha\in M}1^\alpha)')=a>0$ and we can define a σ-additive measure μ: $\exp(I-M)\to\langle 0,a\rangle$ by putting, for any $P\subset I-M$, $\mu(P)=m(\vee_{\alpha\in P}1^\alpha)$. Since $I-M$ must be non-real-measurable, there exists an $\alpha\in I-M$ such that $\mu(\{\alpha\})\neq 0$ and we have derived a contradiction. Thus $\Sigma_{\alpha\in M}m(1^\alpha)=1$. Let us define, for any $\alpha\in M$, a mapping m_α: $L_\alpha\to\langle 0,1\rangle$ by setting

$$m_\alpha(e)=\frac{m(e^\alpha)}{m(1^\alpha)}$$

Then clearly $m_\alpha\in\mathbb{S}(L_\alpha)$ and we have $m(k)=\Sigma_{\alpha\in M}m(1^\alpha)\cdot m_\alpha(k_\alpha)$. The latter equality clearly holds true because $m(\vee_{\alpha\in I-M}k_\alpha)\leqslant m(\vee_{\alpha\in I-M}1^\alpha)=0$. We have obtained the desired expression of m and the proof is finished. ■

Remark 2. Note that the set-theoretic assumption in Theorem 2 can not be omitted. Suppose I is real-measurable and $L_\alpha=\{0,1\}$ for any $\alpha\in I$. Let μ be the measure which makes I real-measurable. Define m: $\Pi_{\alpha\in I}L_\alpha\to\langle 0,1\rangle$ by setting $m(e)=\mu(\{\alpha|e_\alpha=1\})$. Then m is a state on $\Pi_{\alpha\in I}L_\alpha$ which does not depend on countably many coordinates.

Let us derive two consequences of the latter theorem. Recall firstly that a state m is called pure if the equality $m=cm_1+(1-c)m_2$, $0<c<1$, implies $m=m_1=m_2$.

Corollary 1. Let $L=\Pi_{\alpha\in I}L_\alpha$ be a product of logics, I a set of non-real-measurable cardinality. Then L is without pure states iff all L_α are without pure states.

Proof. The proof is an easy consequence of Theorem 2. Observe that Corollary 1 is no longer valid as soon as there are real-measurable cardinals (Remark 2 and Theorem 6.6 in Varadarajan, 1968). ■

The second corollary concerns logics with discrete centers. The center of a logic L is the set $C(L)=\{a\in L \mid a$ is compatible with all $b\in L\}$. It is known that $C(L)$ is a sub-σ-algebra of L (Varadarajan, 1968; Maeda and Maeda, 1970). We say that L is a logic with discrete center if $C(L)$ is isomorphic to the σ-algebra of all subsets of a set.

Before stating the second corollary, we need a generalized version of Theorem 2.14 in Varadarajan (1968). We shall deal with a certain class of logics.

Definition 5. Suppose τ is a cardinal. We say that a logic is a τ-logic if the following statements on L hold:

(i) L is a logic with discrete center and the cardinality of the set of all atoms in $C(L)$ is not greater than τ.

(ii) If $L'\subset L$, card $L'\leqslant\tau$ and any two elements of L' are orthogonal, then L' has the least upper bound in L.

Denote, for an element $a\in L$, $S_a=\{h\in L \mid h\leqslant a\}$.

Proposition 2. Suppose L is a τ-logic. If we put $A=\{a \mid a$ is an atom in $C(L)\}$, then L is isomorphic to $\prod_{a\in A}S_a$. The proof of the latter proposition can be modeled on the proof of Theorem 2.14 in Varadarajan (1968). The isomorphism i: $L\to\prod S_a$ maps $h\in L$ to the point of $\prod_{a\in A}S_a$ whose ath coordinate is $h\wedge a$.

Definition 6. We say that a state m on a logic L is carried by an element $h\in L$ if $m(h')=0$.

> *Corollary 2.* Suppose L is a τ-logic, τ is non-real-measurable. (1) Any state on L is a strong convex combination of the states which are carried by atoms of $C(L)$. (2) If there is no pure state on any S_a, a an atom of $C(L)$, then there is no pure state on L.

The proof follows immediately from Proposition 2 and Theorem 2.

Let us comment in concluding that our Corollaries overlap somewhat with a result of Varadarajan. Namely, it can be shown easily that if I is countable then Corollary 1 follows from Theorem 6.19 in Varadarajan (1968) and so does Corollary 2.2 for τ countable. Our results may therefore be viewed as generalizations of the quoted theorem to higher cardinals (or maybe to all cardinals). In our opinion, part 1 of Corollary 2 is new even for the case when the center has a countable number of atoms.

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